On the Integrals in Series of Binomial Differential Equations.

By William Woolsey Johnson.

1. The term "binomial equation" was applied by Boole to the linear equation whose first member consists of two groups of terms, the terms of each group being homogeneous. Employing the notation

$$x\frac{d}{dx} = \vartheta, \text{ whence } x^2\frac{d^2}{dx^2} = \vartheta(\vartheta - 1), \text{ etc.,}$$

the binomial equation may be written in the form

$$f(\vartheta)\,y - x^s\phi(\vartheta)\,y = X, \tag{1}$$

where f and ϕ are rational integral functions.

If the equation is of the second order, one at least of these functions is of the second degree, and the other of a degree not higher. Supposing both to be of the second degree, the binomial equation of the second order may be written

$$(\vartheta - a)(\vartheta - b)\,y - qx^s(\vartheta - c)(\vartheta - d) = X. \tag{2}$$

2. In integration by series it has been usual to consider only the case in which the second member is zero, in which case the substitution

$$y = \sum_0^\infty A_r x^{m+rs}$$

gives (since $\vartheta x^t = tx^t$),

$$\sum_0^\infty A_r\{(m+rs-a)(m+rs-b)\,x^{m+rs} - q\,(m+rs-c)(m+rs-d)\,x^{m+(r+1)s}\} = 0,$$

which is satisfied if $\qquad A_0(m-a)(m-b) = 0 \qquad (3)$

and, when $r > 0$,

$$(m+rs-a)(m+rs-b)\,A_r - q\,(m+\overline{r-1}\,s-c)(m+\overline{r-1}\,s-d)\,A_{r-1} = 0. \tag{4}$$

Equation (3) gives two values of m, and equation (4) determines the ratio

between consecutive coefficients. Thus we have in general two independent integrals of the form $\sum_0^\infty A_r x^{a+rs}$ and $\sum_0^\infty B_r x^{b+rs}$ respectively. These series are convergent when $x<1$, if s is positive, and convergent when $x>1$, if s is negative. In like manner, we can obtain two series of the form $\sum_0^\infty A_r x^{c-rs}$ and $\sum_0^\infty B_r x^{d-rs}$, and these series will be convergent when the former series are divergent.

But if one of the functions f and ϕ is of the first degree, it is necessary to take for $f(\vartheta)$ that which is of the second degree, and we obtain two integrals of the form $\Sigma A_r x^{m+rs}$ which are always convergent, since by equation (4) the ratio $A_r : A_{r-1}$ will in this case contain two factors in the denominator which increase without limit, and only one such factor in the numerator. We can also obtain one series in the form Σx^{m-rs}, but it will be divergent for all values of x. In like manner, if the function ϕ is a mere constant, we have in general two integrals in the form of series convergent for all values of x.

It is noticeable that the trinomial equation of the second order does not thus always admit of integrals in the form of convergent series in x. Thus the equation being

$$f(\vartheta)\,y + x\phi(\vartheta)\,y + x^2\chi(\vartheta)\,y = 0,$$

the function $\phi(\vartheta)$ may be the only one of the second degree, and in that case we should have only divergent series either in ascending or in descending powers of x.

3. In discussing the binomial equation of the second order we may, without loss of generality, take q and s in equation (2) each equal to unity. For if we put

$$z = qx^s, \text{ whence } \vartheta' = z\frac{d}{dz} = \frac{\vartheta}{s},$$

the equation becomes

$$\left(\vartheta' - \frac{a}{s}\right)\left(\vartheta' - \frac{b}{s}\right)y - z\left(\vartheta' - \frac{c}{s}\right)\left(\vartheta' - \frac{d}{s}\right) = Z.$$

Taking therefore

$$(\vartheta - a)(\vartheta - b)\,y - x(\vartheta - c)(\vartheta - d)\,y = 0 \tag{5}$$

as the standard form when the second member is zero, the relation between consecutive coefficients is, from equation (4),

$$A_r = \frac{(m+r-1-c)(m+r-1-d)}{(m+r-a)(m+r-b)} A_{r-1}, \tag{6}$$

and, writing the complete integral in the form $y = A_0 y_1 + B_0 y_2$, we have the two independent integrals

$$\left.\begin{aligned} y_1 &= x^a\left[1 + \frac{(a-c)(a-d)}{1(a-b+1)}x + \frac{(a-c)(a-c+1)(a-d)(a-d+1)}{1.2(a-b+1)(a-b+2)}x^2 + \dots\right] \\ &\text{and} \\ y_2 &= x^b\left[1 + \frac{(b-c)(b-d)}{1(b-a+1)}x + \frac{(b-c)(b-c+1)(b-d)(b-d+1)}{1.2(b-a+1)(b-a+2)}x^2 + \dots\right], \end{aligned}\right\} \tag{7}$$

or, in the notation of the hypergeometric series,

$$y_1 = x^a F(\alpha, \beta, \gamma, x),$$
$$y_2 = x^b F(\alpha', \beta', \gamma', x),$$

where

$$\alpha = a - c, \qquad \alpha' = b - c,$$
$$\beta = a - d, \qquad \beta' = b - d,$$
$$\gamma = a - b + 1, \quad \gamma' = b - a + 1.$$

4. If we further transform equation (5) by putting $y = x^a v$, we have, since $f(\vartheta)\, x^a v = x^a f(\vartheta + a)\, v$,

$$\vartheta(\vartheta + a - b)\, v - x(\vartheta + a - c)(\vartheta + a - d)\, v = 0,$$

or

$$\vartheta(\vartheta + \gamma - 1)\, v - x(\vartheta + \alpha)(\vartheta + \beta)\, v = 0,$$

which is identical with the differential equation of the hypergeometric series as derived by Gauss from the properties of the series itself (Werke, Band III, p. 207). The binomial equation (2) is therefore reduced to Gauss' equation by the transformations $z = qx^s$, $y = z^{\frac{a}{s}} v$.

If the function ϕ in equation (1) is of the first degree, say $\phi(\vartheta) = q(\vartheta - c)$, q and s are each reduced to unity by putting x in place of $\frac{qx^s}{s}$, and we have then only to supply the factor $\frac{\vartheta - d}{-d}$ where d is infinite, to make the above general solution applicable; this is plainly equivalent to omitting the d-factors in equations (7). We have in this case

$$y_1 = x^a F\left(\alpha, \beta, \gamma, \frac{x}{\beta}\right),$$

where $\beta = \infty$, and x now stands for $\frac{qx^s}{s}$. In like manner, when $\phi(\vartheta) = q$, we omit also the c-factors, and we have

$$y_1 = x^a F\left(\alpha,\ \beta,\ \gamma,\ \frac{x}{\alpha\beta}\right),$$

where $\alpha = \infty$, $\beta = \infty$, and x stands for $\frac{qx^s}{s^2}$.

5. If one of the differences $a - c$, $a - d$, $b - c$, $b - d$ is zero or a negative integer, y_1 or y_2 becomes a finite series.

If $a = b$ the two series become identical, and if a and b differ by an integer, one of the integrals becomes infinite, a zero factor occurring in a denominator. We may in this last case assume a to be the greater of the two roots of f, so that $\gamma > 1$ and y_2 is the integral which fails. I have given elsewhere* the second integral in the case $\gamma \overline{\gtrless} 1$, in the form

$$y_2 = y_1 \log x + (-1)^\gamma \frac{(\gamma - 2)!\ (\gamma - 1)!}{(\alpha + 1 - \gamma) \ldots . (\alpha - 1)(\beta + 1 - \gamma) \ldots . (\beta - 1)} t + y', \quad (8)$$

where t is the sum of the terms of y_2 in equation (7) which precede the first infinite term (when $\gamma = 1$, $t = 0$), and

$$y' = x^a \left[\frac{\alpha\beta}{1.\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{1} - \frac{1}{\gamma}\right) x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}\right.$$
$$\left. \times \left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{1} - \frac{1}{2} - \frac{1}{\gamma} - \frac{1}{\gamma+1}\right) x^2 + \ldots .\right], \quad (9)$$

a series differing from y_1 only in that each coefficient is multiplied by a quantity which may be called its *adjunct*, consisting of the sum of the reciprocals of the factors in the numerator taken positively and of those in the denominator taken negatively. The cases were also considered in which $a - b$ is an integer and at the same time one of the differences $a - c$, $a - d$ or $b - c$, $b - d$ is zero or a negative integer.

In the present paper the same method will be applied to the analogous special cases which arise in connection with the particular integral of the equation

$$(\vartheta - a)(\vartheta - b)\, y - x\,(\vartheta - c)(\vartheta - d)\, y = kx^p, \quad (10)$$

and the results will in fact include those previously found concerning the integrals y_1 and y_2 when the second member is zero.

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* The Messenger of Mathematics, Vol. XVI, p. 35, July, 1887.

6. Writing the complete integral of equation (10) in the form

$$y = A_0 y_1 + B_0 y_2 + Y,$$

y_1 and y_2 are the series given in equations (7), and putting $Y = \sum_0^\infty C_r x^{m+r}$, we have, instead of equation (3),

$$C_0 (m-a)(m-b)\, x^m = kx^p,$$

whence $m = p$, and
$$C_0 = \frac{k}{(p-a)(p-b)}\,.$$

The relation between consecutive coefficients is still given by equation (6) and the result is

$$Y = \frac{kx^p}{(p-a)(p-b)}\Big[1 + \frac{(p-c)(p-d)}{(p-a+1)(p-b+1)}\,x + \frac{(p-c)(p-c+1)(p-d)(p-d+1)}{(p-a+1)(p-a+2)(p-b+1)(p-b+2)}\,x^2 + \ldots\Big]. \quad (11)$$

If in the differential equation (10) we put $k = 0$, the exponent p is arbitrary; the value of Y in general reduces to zero; but, if we suppose $p = a$ or $p = b$, we have the product of an indeterminate coefficient into y_1 or y_2 as given in equations (7).

7. Among the special cases which arise we notice first that in which one of the differences $p - c$ or $p - d$ is zero or a negative integer. Y is in this case a finite series.

In the next place suppose one of the differences $p - a$ or $p - b$ to be a positive integer, say $p - a = i$. Then the $(i+1)^{\text{th}}$ term of y_1 contains the same power of x that occurs in the first term of Y. Let M be a constant such that the $(i+1)^{\text{th}}$ term of My_1 is identical with the first term of Y, then all the following terms will be identical, and the particular integral $Y' = Y - My_1$ will be a finite expression containing i terms. This finite integral would of course have been obtained directly if we had employed a descending series, in which case we should have written the differential equation in the form

$$(\vartheta - c)(\vartheta - d)\, y - x^{-1}(\vartheta - a)(\vartheta - b)\, y = -kx^{p-1},$$

and the particular integral would be

$$Y' = \frac{-kx^{p-1}}{(p-c-1)(p-d-1)}\Big[1 + \frac{(p-a-1)(p-b-1)}{(p-c-2)(p-d-2)}\,x^{-1} + \ldots\ldots\Big].$$

8. If $p-a$ or $p-b$ is zero or a negative integer, Y in equation (11) is infinite; thus, if $p-a=-i$, the coefficient of x^{p+i} contains a zero factor in the denominator. To find a particular integral having a finite value, we may put
$$p=a-i+h,$$
in which h will ultimately be put equal to zero. Denoting the sum of the first i terms by T, the value of Y is
$$Y=T+k\frac{(p-c)\ldots\ldots(p-c+i-1)(p-d)\ldots\ldots(p-d+i-1)}{(p-a)\ldots\ldots(p-a+i)(p-b)\ldots\ldots(p-b+i)}x^{p+i}$$
$$\times\left[1+\frac{(p-c+i)(p-d+i)}{(p-a+i+1)(p-b+i+1)}x+\ldots\ldots\right],$$
or
$$Y=T+k\frac{(p-c)\ldots\ldots(a-c-1+h)(p-d)\ldots\ldots(a-d-1+h)}{(-i+h)\ldots\ldots(-1+h)\,h\,(p-b)\ldots\ldots(a-b+h)}x^{a+h}$$
$$\times\left[1+\frac{(a-c+h)(a-d+h)}{(1+h)(a-b+1+h)}x+\ldots\ldots\right]. \qquad (12)$$
Denoting the coefficient of x^{a+h} by $\dfrac{M}{h}$, and putting
$$\psi(h)=x^a\left[1+\frac{(a-c+h)(a-d+h)}{(1+h)(a-b+1+h)}x\right.$$
$$\left.+\frac{(a-c+h)(a-c+1+h)(a-d+h)(a-d+1+h)}{(1+h)(2+h)(a-b+1+h)(a-b+2+h)}x^2+\ldots\ldots\right], \qquad (13)$$
the value of Y becomes
$$Y=T+\frac{M}{h}(1+h\log x+\ldots\ldots)[y_1+h\psi'(0)+\ldots\ldots], \qquad (14)$$
since, by the first of equations (7), $y_1=\psi(0)$. Therefore, putting Y' for the particular integral $Y-\dfrac{M}{h}y_1$, we have
$$Y'=T+My_1\log x+M\psi'(0)+\ldots\ldots,$$
the terms not written involving powers of h.

When we put $h=0$, T in this equation is the sum of the terms in equation (11) which precede the first infinite term, M is the coefficient of this term with the zero factor in the denominator omitted, and, denoting $\psi'(0)$ by y', we have for the integral when $p=a-i$,
$$Y'=T+My_1\log x+My'. \qquad (15)$$

In order to expand y' in powers of x, let us write
$$\psi(h)=\sum_0^\infty H_r x^{a+r};$$

so that, by equation (13),

$$H_0 = 1 \text{ and } H_r = \frac{(a-c+h)\ldots\ldots(a-c+r-1+h)(a-d+h)\ldots\ldots(a-d+r-1+h)}{(1+h)\ldots\ldots(r+h)(a-b+1+h)(a-b+r+h)};$$

then

$$\psi(h) = \sum_0^\infty \frac{dH_r}{dh}x^{a+r} = \sum_1^\infty H_r \frac{d\log H_r}{dh}x^{a+r},$$

hence

$$y' = \psi(0) = x^a\Big[\frac{(a-c)(a-d)}{1(a-b+1)}\Big(\frac{1}{a-c}+\frac{1}{a-d}-\frac{1}{1}-\frac{1}{a-b+1}\Big)x$$
$$+\frac{(a-c)(a-c+1)(a-d)(a-d+1)}{1.2(a-b+1)(a-b+2)}\Big(\frac{1}{a-c}+\frac{1}{a-c+1}+\frac{1}{a-d}$$
$$+\frac{1}{a-d+1}-\frac{1}{1}-\frac{1}{2}-\frac{1}{a-b+1}-\frac{1}{a-b+2}\Big)x^2\ldots\Big], \quad (16)$$

which is identical with equation (9), being formed from the value of y_1, equations (7), by the law of the adjuncts mentioned in §5.

9. If we put $k=0$ and $p=b=a-i$, equation (15) gives the value of y_2 when a and b differ by an integer. Writing it in the form

$$Y' = M\Big(y_1\log x + y' + \frac{T}{M}\Big),$$

and recurring to the values of T and M, see equations (11) and (12), we find that M is now indeterminate, but

$$\frac{T}{M} = -(-1)^i\frac{i!\,(i-1)!}{(b-c)\ldots\ldots(b-c+i-1)(b-d)\ldots\ldots(b-d+i-1)}x^b$$
$$\times\Big[1+\frac{(b-c)(b-d)}{1(b-a+1)}x+\ldots\ldots\Big];$$

we may therefore put, when $b=a-i$,

$$y_2 = y_1\log x + y' - (-1)^i\frac{i!\,(i-1)!}{(b-c)\ldots\ldots(a-c-1)(b-d)\ldots\ldots(a-d-1)}t, \quad (17)$$

where t stands for the sum of the terms which precede the first infinite term in y_2 as written in equations (7). Equation (17) is equivalent to the equation (8) quoted from a preceding paper.

10. The integral in equation (15) may be written out in accordance with the following rule: Write the terms as in equation (11) until a coefficient with a zero factor in the denominator is reached. Express the remainder of the integral as the product of this coefficient and a series whose first coefficient is unity. Omit the zero factor in the coefficient, and multiply each term in the series by the sum of log x and the adjunct of its coefficient.

11. Let us next suppose that, while a zero factor occurs in the denominator of a coefficient, one also occurs in the numerator of a coefficient; for example, suppose that $p-c$ as well as $p-a$ is zero or a negative integer. If the zero factor in the numerator occurs in the same or in an earlier coefficient than that in the denominator, no infinite coefficient occurs. Denoting the sum of the terms preceding that in which the zero factor occurs in the numerator by Y', Y is the sum of Y' and an indeterminate multiple of y_1, hence Y' is a particular integral. But if the zero factor first appears in the denominator, the integral is given by equation (15), and y_1 is a finite series. It is, however, to be noticed that y' is still an infinite series, for the adjunct of each of the vanishing coefficients in y_1 contains the reciprocal of the zero factor, and therefore simply destroys the zero factor, so that the remaining terms of y' are the vanishing terms of y_1 with the zero factor omitted. If we had regarded $\psi(h)$ as standing only for those terms in the second member of equation (13) which correspond to existing terms in y_1, we should have written $\psi(h)+h\chi(h)$ for the first member, and the final factor in equation (14) would have been

$$y_1+h\psi'(0)+h\chi(0)+\ldots.$$

Equation (13) would then be obtained by putting $y'=\psi'(0)+\chi(0)$ in which $\psi'(0)$ now represents the existing terms of y_1 with adjuncts, and $\chi(0)$ the vanishing terms with the zero factor omitted. These terms in My' are the corresponding terms as they occur in Y, equation (11), with both zero factors omitted. Thus the rule for writing the integral given in §10 is in this case to be thus supplemented: If a zero factor occurs subsequently in a numerator, omit it, and discontinue the use of the logarithm and adjuncts.

12. Next, let us suppose that each of the differences $p-a$ and $p-b$ is zero or a negative integer; in this case two zero factors will occur in the denominators in equation (11). Of the two numbers a and b, let a be that which is not less than the other, we may then put

$$b=a-i \text{ and } p=b-j,$$

where i and j are positive integers or zero, and the coefficient of x^{p+j} is the first which is infinite. As before, we change the value of p to $b-j+h$, and ultimately put $h=0$.

Denoting, as in §8, by T the sum of the preceding terms, and by $\dfrac{M}{h}$ the

coefficient which is infinite when $h = 0$, the value of Y becomes

$$Y = T + \frac{M}{h} x^{p+j}\left[1 + \frac{(p-c+j)(p-d+j)}{(p-a+j+1)(p-b+j+1)} x + \ldots\right],$$

or

$$Y = T + \frac{M}{h} x^h . x^b \left[1 + \frac{(b-c+h)(b-d+h)}{(b-a+1+h)(1+h)} x + \ldots \right.$$

$$\left. + \frac{(b-c+h)\ldots(a-c-1+h)(b-d+h)\ldots(a-d-1+h)}{(-i+1+h)\ldots h(1+h)\ldots(i+h)} x^i \left\{1 + \frac{(a-c+h)(a-d+h)}{(1+h)(i+1+h)} + \ldots\right\}\right].$$

If we write this in the form

$$Y = T + \frac{M}{h} x^h \left[t + \frac{\eta}{h}\right], \tag{18}$$

t is a function of h which, when $h = 0$, becomes identical with t in equation (17) and

$$\eta = N\psi(h), \tag{19}$$

where $\psi(h)$ is the function defined in equation (13). Employing suffixes to denote the values assumed when $h = 0$, we have

$$\eta_0 = N_0 \psi(0) = N_0 y_1,$$

and it is readily seen from equation (17) that the complementary function in this case is

$$Ay_1 + B\left[y_1 \log x + y' + \frac{t_0}{N_0}\right].$$

Using accents to indicate differentiation with reference to h, equation (18) gives

$$Y = T + \frac{M}{h}(1 + h\log x + \ldots)(t_0 + ht_0' + \ldots)$$

$$+ \frac{M}{h^2}\left(1 + h\log x + \frac{1}{2}h^2\log^2 x + \ldots\right)\left(\eta_0 + h\eta_0' + \frac{1}{2}h^2\eta_0'' + \ldots\right),$$

or

$$Y = T + \frac{M}{h^2}\eta_0 + \frac{M}{h}(t_0 + \eta_0\log x + \eta_0')$$

$$+ M\left(t_0\log x + t_0' + \frac{1}{2}\eta_0\log^2 x + \eta_0'\log x + \frac{1}{2}\eta_0''\right) + \ldots,$$

the terms not written involving positive powers of h. From equation (19),

$$\eta_0' = N_0\psi'(0) + \psi(0)\left.\frac{dN}{dh}\right]_0 = N_0 y' + y_1 \left.\frac{dN}{dh}\right]_0;$$

hence the terms involving negative powers of h are

$$\frac{MN_0}{h^2} y_1 + \frac{MN_0}{h}\left[\frac{t_0}{N_0} + y_1\log x + y'\right] + \frac{M}{h}\left.\frac{dN}{dh}\right]_0 y_1.$$

These terms coalesce with the complementary function above; hence, rejecting them and then putting $h = 0$, we may (now omitting the suffixes) write for the particular integral

$$Y' = T + M\left(t \log x + t' + \frac{1}{2}\eta \log^2 x + \eta' \log x + \frac{1}{2}\eta''\right). \tag{20}$$

13. The series denoted by t' and η' are derived from t and η in the same manner that y' in §8 is derived from y_1, namely, by multiplying each coefficient by its adjunct. To find η'', let

$$\eta = \sum_0^\infty H_r x^{a+r},$$

then each H is of the form

$$\frac{(\alpha + h)(\beta + h)\ldots\ldots}{(\lambda + h)(\mu + h)\ldots\ldots}.$$

Differentiating

$$\frac{d\eta}{dh} = \sum_0^\infty H_r \frac{d \log H_r}{dh} x^{a+r} = \sum_0^\infty H_r \left[\sum \frac{1}{\alpha + h} - \sum \frac{1}{\lambda + h}\right] x^{a+r},$$

and

$$\frac{d^2\eta}{dh^2} = \sum_0^\infty H_r \left[\left\{\sum \frac{1}{\alpha + h} - \sum \frac{1}{\lambda + h}\right\}^2 - \sum \frac{1}{(\alpha + h)^2} + \sum \frac{1}{(\lambda + h)^2}\right] x^{a+r};$$

whence, putting $h = 0$,

$$\eta' = \sum_0^\infty H_r \left[\sum \frac{1}{\alpha} - \sum \frac{1}{\lambda}\right] x^{a+r} = \sum_0^\infty H_r\,(\text{adj})\, x^{a+r},$$

and

$$\eta'' = \sum_0^\infty H_r \left[(\text{adj})^2 - \sum \frac{1}{\alpha^2} + \sum \frac{1}{\lambda^2}\right] x^{a+r}. \tag{21}$$

Thus η'' in equation (20) is the same series as that represented by η except that each coefficient is multiplied by a factor which may be called its *secondary adjunct*, which consists of the square of the adjunct together with the sum of the squares of the reciprocals of the factors in the denominator taken positively and of those in the numerator taken negatively.

14. The initial terms of equation (20) show that the integral may be written out in accordance with the rule given in §10 until the second zero factor is reached, and the final terms give the additional rule: After a second zero factor in a denominator is reached, omit this factor and multiply each term by the

sum of $\frac{1}{2}\log^2 x$, the product of $\log x$ by the adjunct of the coefficient, and one half of the secondary adjunct.

15. Finally, let us suppose that, while two zero factors occur in the denominators, a zero factor also occurs in a numerator, as in §11. If the occurrence of this factor is subsequent to that of the first but not of the second of those in the denominator, the particular integral ceases to take the form (20). In this case y_1 will be a finite series and y_2 will be of the regular form (7), containing higher powers of x than any which occur in y_1. The particular integral will be of the form (15) except that the use of the logarithm and adjuncts is discontinued in the interval between the appearance of the zero factor in the numerator and the second of those in the denominator; in other words, in the case of those powers of x which do not occur in either term of the complementary function.

16. If, however, the zero factor in the numerator occurs subsequently to both of those in the denominator, the particular integral is given by equation (20), but a factor α in one of the coefficients H and in every subsequent coefficient vanishes. The secondary adjunct in equation (21) may be written

$$\left(\frac{1}{\alpha}+\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)^2-\frac{1}{\alpha^2}-\sum\frac{1}{\beta^2}+\sum\frac{1}{\lambda^2},$$

or
$$2\frac{1}{\alpha}\left(\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)+\sum\left(\frac{1}{\beta}-\sum\frac{1}{\lambda}\right)^2-\sum\frac{1}{\beta^2}+\sum\frac{1}{\lambda^2}.$$

Hence one half the secondary adjunct is the infinite quantity

$$\frac{1}{\alpha}\left(\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right),$$

and, since the adjunct itself is the infinite quantity $\frac{1}{\alpha}$, the factor by which every vanishing coefficient in η is multiplied is of the form

$$\frac{1}{\alpha}\left[\log x+\sum\frac{1}{\beta}-\sum\frac{1}{\lambda}\right].$$

The effect, therefore, is to cancel the vanishing factor and to multiply the term of η by the sum of $\log x$ and the adjunct of its coefficient as it now stands. This result was to have been expected, since, regarding the zero factor as cancelling one of those which have already occurred in the denominator, the subsequent terms may be regarded as belonging to the series t. In like manner, if a second zero factor occurs in a numerator (all four of the differences $p-a$, $p-b$, $p-c$ and $p-d$ being zero or negative integers), the terms return to the condition of

the series T in equation (20), and are written as in equation (11), the four zero factors cancelling one another.

17. The complete rule for writing out the particular integral may be expressed as follows:

Write out the terms as in equation (11); let the series terminate if a zero coefficient occurs; in the infinite coefficients (which belong to powers of x occurring in one term of the complementary function), omit the zero factors and multiply by the sum of the logarithm and adjunct; and in the doubly infinite coefficients (which belong to powers of x occurring in both terms of the complementary function, in one of them with logarithms and adjuncts) multiply by the sum of the half square of the logarithm, the product of logarithm and adjunct, and the half secondary adjunct as in §14.

It is to be noticed that in forming the adjuncts and secondary adjuncts in equation (20), we ignore the factors included in the coefficient M. To include these factors would not indeed be incorrect, but would merely be equivalent to adding to Y' a quantity included in the complementary function.

18. In what precedes, the binomial equation is taken to be of the second order. The results are obviously similar when the equation is of the first order. Thus, supposing it reduced as in §3 to the standard form

$$(\vartheta - a)\, y - x\,(\vartheta - c)\, y = X,$$

the complementary function is

$$y_1 = x^a \left[1 + \frac{a-c}{1}\, x + \frac{(a-c)(a-c+1)}{1.2}\, x^2 + \ldots.\right]$$

[in finite form $y_1 = x^a\,(1-x)^{c-a}$]; and if $X = kx^p$, the particular integral is

$$Y = \frac{kx^p}{p-a}\left[1 + \frac{p-c}{p-a+1}\, x + \frac{(p-c)(p-c+1)}{(p-a+1)(p-a+2)}\, x^2 + \ldots.\right].$$

But if $p - a$ is zero or a negative integer, an infinite coefficient occurs and we have the form

$$Y' = T + My_1 \log x + My',$$

as in equation (15), so that the rule for writing out the result is precisely the same as in §10.

19. It may also be noticed that the values of y_1 and Y above are the independent integrals of a special case of the equation of the second order when the second member is zero; for they are equivalent to the results of putting

$d = b - 1$ in equations (7). The case is that in which the equation admits of an integrating factor of the form x^m. Thus multiplying

$$(\vartheta - a)(\vartheta - b)\, y - x\,(\vartheta - c)(\vartheta - b + 1)\, y = 0$$

by x^{-b-1}, we have

$$x^{-b-1}(\vartheta - b)(\vartheta - a)\, y - x^{-b}(\vartheta - b + 1)(\vartheta - c)\, y = 0$$

which, by the theorem (C), p. 96 of the preceding volume of this Journal, may be written $$D\left[x^{-b}(\vartheta - a)\, y - x^{-b+1}(\vartheta - c)\, y\right] = 0,$$

whence, integrating and multiplying by x^b, we have

$$(\vartheta - a)\, y - x\,(\vartheta - c)\, y = kx^b,$$

which is the binomial equation of the first order, its complementary function giving one, and its particular integral (involving k, which is now a constant of integration) furnishing the other of the two independent integrals of the given equation.

In like manner, the three series in equations (7) and (11) are the three independent integrals of a special case of the binomial equation of the third order.

20. The particular integral of the general binomial equation of the third order, when the second member is a power of x, is obviously of a form similar to Y in equation (11); thus, if the equation is reduced to the standard form

$$(\vartheta - a)(\vartheta - b)(\vartheta - c)\, y - x\,(\vartheta - \alpha)(\vartheta - \beta)(\vartheta - \gamma)\, y = kx^p,$$

the particular integral is

$$Y = \frac{kx^p}{(p-a)(p-b)(p-c)}\left[1 + \frac{(p-\alpha)(p-\beta)(p-\gamma)}{(p-a+1)(p-b+1)(p-c+1)}\, x + \ldots\right]. \quad (22)$$

The same rules also apply with reference to those terms whose coefficients, as written in accordance with this equation, would be infinite or doubly infinite. In fact, the demonstrations already given are applicable to binomial equations of any order.

21. For a term which would be triply infinite, the same method gives a similar extension of the rule. Thus, in place of equation (18), we have

$$Y = T + \frac{M}{h}\, x^h\left[t + \frac{\tau}{h} + \frac{\eta}{h^2}\right],$$

in which T, t, τ and η denote groups of terms written as in equation (22) except that the zero factors are omitted, there having been in the groups t, τ and η, an

excess of one, two and three zero factors respectively in the denominator of each coefficient. In the development of this equation

$$
\begin{aligned}
Y = T &+ \frac{M}{h}(1 + h\log x + \ldots.)(t + ht' + \ldots.) \\
&+ \frac{M}{h^2}\left(1 + h\log x + \frac{1}{2}h^2\log^2 x + \ldots.\right)\left(\tau + h\tau' + \frac{1}{2}h^2\tau'' + \ldots.\right) \\
&+ \frac{M}{h^3}\left(1 + h\log x + \frac{1}{2}h^2\log^2 x + \frac{1}{6}h^3\log^3 x + \ldots.\right) \\
&\quad \times \left(\eta + h\eta' + \frac{1}{2}h^2\eta'' + \frac{1}{6}h^3\eta''' + \ldots.\right),
\end{aligned}
$$

the terms involving negative powers of h will be found to coalesce with the complementary function, so that the particular integral corresponding to equation (20) is the coefficient of h^0 in the development, that is

$$
\begin{aligned}
Y' = T + M\Big(t\log x + t' &+ \frac{1}{2}\tau\log^2 x + \tau'\log x + \frac{1}{2}\tau'' \\
&+ \frac{1}{6}\eta\log^3 x + \frac{1}{2}\eta'\log^2 x + \frac{1}{2}\eta''\log x + \frac{1}{6}\eta'''\Big). \quad (23)
\end{aligned}
$$

This equation shows that the rules already given still apply to the terms which, as first written, are singly and doubly infinite, and for those which are triply infinite gives the rule: Multiply each coefficient by $\frac{1}{6}\log^3 x$, the product of $\frac{1}{2}\log^2 x$ by the adjunct, the product of $\frac{1}{2}\log x$ by the secondary adjunct and one sixth of a *tertiary adjunct.*

22. The tertiary adjunct of a coefficient in η is the factor by which the coefficient must be multiplied to produce the corresponding coefficient of η'''. Employing the notation of §13 for any coefficient, thus

$$H = \frac{\alpha\beta\gamma\ldots.}{\lambda\mu\nu\ldots.},$$

let

$$
\begin{aligned}
S_1 &= \sum\frac{1}{\alpha} - \sum\frac{1}{\lambda}, \\
S_2 &= \sum\frac{1}{\lambda^2} - \sum\frac{1}{\alpha^2}, \\
S_3 &= \sum\frac{1}{\alpha^3} - \sum\frac{1}{\lambda^3}, \\
&\ldots\ldots\ldots\ldots\ldots \\
S_n &= (-1)^{n-1}\left[\sum\frac{1}{\alpha^n} - \sum\frac{1}{\lambda^n}\right].
\end{aligned}
$$

Denoting by D the operation of taking the derivative with respect to a quantity h, added to each of the quantities $\alpha, \beta, \gamma, \ldots$ and $\lambda, \mu, \nu, \ldots$ and afterwards put equal to zero,

$$DS_1 = S_2,\ DS_2 = 2S_3,\ DS_3 = 3S_4, \ldots DS_n = nS_{n+1}.$$

We have then

$$\begin{aligned}
DH &= HS_1,\\
D^2H &= H(S_1^2 + S_2).\\
D^3H &= H(S_1^3 + 3S_1S_2 + 2S_3),\\
D^4H &= H(S_1^4 + 6S_1^2S_2 + 8S_1S_3 + 6S_4 + 3S_2^2),\\
D^5H &= H(S_1^5 + 10S_1^3S_2 + 20S_1^2S_3 + 30S_1S_4 + 24S_5 + 15S_1S_2^2 + 20S_2S_3),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

for the corresponding coefficients in $\eta', \eta'', \eta''', \ldots$; thus, for example, the tertiary adjunct is $S_1^3 + 3S_1S_2 + 2S_3$.

23. In general, if η represent a group of terms in which as first written there is an excess of n zero factors in the denominator of each coefficient, the terms corresponding to η in the integral, derived as in §21, will obviously be the coefficient of h^0 in the development of

$$\frac{M}{h^n} e^{h \log x} . e^{hD} \eta ;$$

that is to say, it is

$$\frac{M}{n!} (\log x + D)^n \eta .$$

24. As an example of the process, let us take the equation

$$(x^3 - x^4) \frac{d^3y}{dx^3} + (9x^2 - 2x^3) \frac{d^2y}{dx^2} + (18x + 2x^2) \frac{dy}{dx} + 6y = kx^{-5},$$

which, written in the ϑ notation, is

$$(\vartheta + 1)(\vartheta + 2)(\vartheta + 3)\, y - x\, (\vartheta - 2)\, \vartheta\, (\vartheta + 1)\, y = kx^{-5}.$$

By equation (22) we have

$$\begin{aligned}
Y = \frac{kx^{-5}}{-4.-3.-2} &+ \text{pr} \frac{-7.-5.-4}{-3.-2.-1} x^{-4} + \text{pr} \frac{-6.-4.-3}{-2.-1.0} x^{-3}\\
&+ \text{pr} \frac{-5.-3.-2}{-1.0.1} x^{-2} + \text{pr} \frac{-4.-2.-1}{0.1.2} x^{-1}\\
&+ \text{pr} \frac{-3.-1.0}{1.2.3} x^0 + \text{pr} \frac{-2.0.1}{2.3.4} x^1 + \text{pr} \frac{-1.1.2}{3.4.5} x^2\\
&+ \text{pr} \frac{0.2.3}{4.5.6} x^3 + \text{pr} \frac{1.3.4}{5.6.7} x^4 + \text{pr} \frac{2.4.5}{6.7.8} x^5 + \ldots,
\end{aligned}$$

in which, for shortness, the character pr in each coefficient is used to denote the preceding coefficient. The coefficient of x^r is here triply infinite only for $r=-1$; it is doubly infinite for $r=-2$ and $r=0$, and singly infinite for $r=-3$, $r=1$ and $r=2$; the remaining terms belong to the group T. It is convenient to calculate beforehand the necessary values of S_1, S_2 and S_3, the values of these quantities being zero for the first infinite coefficient, and each successive value being found by adding to the preceding one the part corresponding to the new factors introduced. The result is given in the following table:

	r	S_1	S_2	S_3
$\frac{-5.-3.-2}{1.-1}$	-2	$-\frac{31}{30}$	$\frac{1439}{900}$	$-\frac{4591}{27000}$
$\frac{-4.-2.-1}{2.1}$	-1	$-\frac{31}{30}-\frac{13}{4}=-\frac{257}{60}$	$\frac{1439}{900}-\frac{1}{16}=\frac{5531}{3600}$	$-\frac{4591}{27000}-\frac{145}{64}=-\frac{526103}{216000}$
$\frac{-3.-1}{3.2.1}$	0	$-\frac{257}{30}-\frac{19}{6}=-\frac{149}{20}$	$\frac{5531}{3600}+\frac{1}{4}=\frac{6431}{3600}$	
$\frac{-2.-1}{4.3.2}$	1	$-\frac{149}{20}-\frac{7}{12}=-\frac{241}{30}$		
$\frac{-1.1.2}{5.4.3}$	2	$-\frac{241}{30}-\frac{17}{60}=-\frac{499}{60}$		

We may now write out the value of Y in accordance with the rules established, thus

$$\begin{aligned} Y' = & -\frac{k}{24}x^{-5}-\frac{35k}{36}x^{-4}+35k\Big[x^{-3}\log x+30x^{-2}\Big\{\frac{1}{2}\log^2 x-\frac{31}{30}\log x \\ & +\frac{1}{2}\Big(\frac{31^2}{30^2}+\frac{1439}{900}\Big)\Big\}-120x^{-1}\Big\{\frac{1}{6}\log^3 x+\frac{1}{2}\log^2 x\Big(-\frac{257}{60}\Big) \\ & +\frac{1}{2}\log x\Big(\frac{257^2}{60^2}+\frac{5531}{3600}\Big)+\frac{1}{6}\Big(-\frac{257^3}{60^3}-3\,\frac{257}{60}\,\frac{5531}{3600}-\frac{526103}{216000}\Big)\Big\} \\ & -60\Big\{\frac{1}{2}\log^2 x-\frac{149}{20}\log x+\frac{1}{2}\Big(\frac{149^2}{20^2}+\frac{6431}{3600}\Big)\Big\}+5x\Big(\log x-\frac{241}{30}\Big) \\ & -\frac{1}{6}x^2\Big(\log x-\frac{499}{60}\Big)-\frac{1}{120}x^3\Big(1+\frac{1.3.4}{5.6.7}x+\frac{1.3.4.2.4.5}{5.6.7.6.7.8}x^2+\ldots\Big)\Big], \end{aligned}$$

or finally

$$\begin{aligned} Y' = & -\frac{k}{24}x^{-5}-\frac{35k}{36}x^{-4}+35k\Big[x^{-3}\log x+15x^{-2}\log^2 x-31x^{-2}\log x+40x^{-2} \\ & -20x^{-1}\log^3 x+257x^{-1}\log^2 x-1193x^{-1}\log x \\ & +\frac{21765097}{1296000}x^{-1}-30\log^2 x+447\log x-\frac{5156}{3} \\ & +5x\log x-\frac{241}{6}x-\frac{1}{6}x^2\log x+\frac{499}{360}x^2 \\ & -\frac{1}{120}x^3-\frac{1}{2100}x^4-\ldots\Big]. \end{aligned}$$

If $Ay_1 + By_2 + Cy_3$ be the complementary function in this example, the values of y_1, y_2 and y_3 may be derived in like manner from the value of Y above, taking the coefficients of y_{-1}, y_{-2} and y_{-3} respectively as unity. We should thus find

$$y_3 = x^{-3} + 30x^{-2}\log x - 60x^{-1}\log^2 x + 390x^{-1}\log x - 630x^{-1} - 60\log x + 385 + 5x - \frac{1}{6}x^2,$$

$$y_2 = x^{-2} - 4x^{-1}\log x - 2,$$

and

$$y_1 = x^{-1}$$

(so that the terms containing x^{-1} in the values of Y' and y_3 may be rejected).

25. The following formulae for the verification of integrals consisting of terms of the form x^r, $x^r\log x$, $x^r\log^2 x$, etc., may be noticed in conclusion. From the definition of the symbol ϑ we have

$$\vartheta x^r V = x^r(\vartheta + r)V, \quad \vartheta^2 x^r V = \vartheta x^r(\vartheta + r)V = x^r(\vartheta + r)^2 V, \text{ etc.}$$

whence $f(\vartheta)$ being an algebraic function,

$$f(\vartheta)x^r V = x^r f(\vartheta + r)V.$$

But since

$$\begin{aligned} &\vartheta 1 = 0, \\ &\vartheta \log x = 1, \qquad \vartheta^2 \log x = 0, \\ &\vartheta \log^2 x = 2\log x, \quad \vartheta^2 \log^2 x = 2, \quad \vartheta^3 \log^2 x = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

we have

$$\begin{aligned} f(\vartheta)1 &= f(0), \\ f(\vartheta)\log x &= f'(0) + f(0)\log x, \\ f(\vartheta)\log^2 x &= f''(0) + 2f'(0)\log x + f(0)\log^2 x, \\ f(\vartheta)\log^3 x &= f'''(0) + 3f''(0)\log x + 3f'(0)\log^2 x + f(0)\log^3 x, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Hence, putting in the formula above $V = \log^n x$, we have for $n = 0, 1, 2, \ldots$

$$\begin{aligned} f(\vartheta)x^r &= x^r f(r), \\ f(\vartheta)x^r\log x &= x^r[f'(r) + f(r)\log x], \\ f(\vartheta)x^r\log^2 x &= x^r[f''(r) + 2f'(r)\log x + f(r)\log^2 x], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and in general,

$$f(\vartheta)x^r\log^n x = \left[\frac{d}{d\vartheta} + \log x\right]^n f(\vartheta)x^r.$$

Now if the differential equation be

$$f(\vartheta)y - x\phi(\vartheta)y = kx^p,$$

the integral, which is of the form

$$Y = \Sigma A_r x^r + \log x \Sigma B_r x^r + \log^2 x \Sigma C_r x^r + \ldots\ldots,$$

will be verified if the result of operating upon it with

$$f(\vartheta) - x\phi(\vartheta)$$

is kx^p. The result of operation is, by the formulae just demonstrated,

$$\begin{aligned}
&\Sigma A_r f(r)\, x^r \;\; - \Sigma A_r \phi(r)\, x^{r+1} \\
+\;& \Sigma B_r f'(r)\, x^r - \Sigma B_r \phi'(r)\, x^{r+1} + [\Sigma B_r f(r)\, x^r - \Sigma B_r \phi(r)\, x^{r+1}] \log x \\
+\;& \Sigma C_r f''(r)\, x^r - \Sigma C_r \phi''(r)\, x^{r+1} + 2\,[\Sigma C_r f'(r)\, x^r - \Sigma C_r \phi'(r)\, x^{r+1}] \log x \\
+\;& [\Sigma C_r f(r)\, x^r \;\; - \Sigma C_r \phi(r)\, x^{r+1}] \log^2 x, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

in which the aggregate of terms independent of $\log x$ must reduce to kx^p, and the coefficient of each power of $\log x$ must separately vanish. The first of these conditions involving all the terms of Y will constitute a good verification; and it appears that the aggregate which should reduce to kx^p consists simply of the products of each term independent of $\log x$ by the value of $f(r) - x\phi(r)$ for that term, of each term in the coefficient of $\log x$ by the value of $f'(r) - x\phi'(r)$, and so on. The example in §24 was verified in this way.

The last of the conditions mentioned above shows that the coefficient of the highest power of $\log x$ in Y is in all cases a multiple of a term of the complementary function.